ON SAINT-VENANT'S PRINCIPLE IN THE DYNAMICS OF BEAMS

(O PRINTSIPE SEN-VENANA V DINAMIKE STERZHNEI)

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Two forms of approximate equations for the dynamic flexure of beams are known: the Bernoulli-Euler equation

$$\frac{\partial^4 w}{\partial x^4}+\frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}=0 \tag{0.1}$$

and the Timoshenko equation

$$\frac{\partial^4 w}{\partial x^4}-\left(\frac{1}{v_1^2}+\frac{1}{v_2^2}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{1}{v_1^2 v_2^2}\frac{\partial^4 w}{\partial t^4}+\frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}=0 \tag{0.2}$$

The latter is usually called a "wave equation". It leads to a finite propagation velocity of suddenly applied loads, and for this reason it is considered essentially more suitable for the solution of transient problems than Equation (0.1), which gives an infinite velocity of propagation of wave fronts (see, for example, [1]).

For beams of circular cross section and for strip-beams taken from a rectangular plate of infinite width, Equation (0.2) may be derived from the general equations of the dynamics of an elastic body [2]. To do this, one represents the solution (*) in the form of a series in powers of the distance of a point from the neutral axis and one disregards all terms and derivatives of higher than fourth order (in time as well as in the coordinates) in the differential equations of infinite order that are obtained.

The fact that in the derivation of (0.2) it was necessary to neglect higher derivatives, would seem to indicate that the application of this equation to problems of transient propagation of deformations along the beam is excluded. To this one should add that (0.1) as well as (0.2) allow one to formulate boundary conditions only in the sense of Saint-Venant.

The sufficiency of the latter is guaranteed in static problems by the Saint-Venant principle. In accordance with this principle, an improvement in the theory by the inclusion of the effects of self-equilibrating boundary loads would lead only to local corrections in the stress field and would not reflect (practically) on the values of deflections. In dinamical problems on the other hand, stresses and displacements excited by an end load (without regard to whether or not it is self-equilibrating) are not limited to a

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*) See likewise the dissertation of I.G. Selezov "The investigation of the propagation of elastic waves in plates and shells". Institute of Mechanics, Academy of Sciences Ukrainian SSR, 1961.

narrow zone near the loaded end, as a result of which the possibility in such problems of posing boundary conditions only "in the sense of Saint-Venant" is problematical.

The above casts doubt on the admissibility of the "wave" equation (0.2) (and even more on the "nonwave" equation (0.1)) for the solution of problems of deformation propagation . The applicability of the equations is doubtful because of the restrictions on their solutions satisfying boundary conditions, and likewise even in cases when the boundary conditions do not contradict the applicability of the equations (i.e. formulated according to Sain-Venant).

There are a number of theoretical [2 to 7].and experimental [5 and 6] papers on the applicability of Equations (0.1) and (0.2); however, until the present time there exist contradictory opinions on this question. In the opinion of some, Equations (0.1) and (0.2) are not suitable for the solution of problems of the propagation of disturbances and may be applied only to the study of processes which change sufficiently smoothly both in time and in the coordinates. In the opinion of others, Equation (0.2) covers all problems in which the boundary conditions are given in the sense of Saint-Venant.

In the present paper we attempt to answer two fundamental questions.

1. To what extent is it valid to carry over the principle of Saint-Venant to the dinamics of beams if one limits the boundary conditions to those of Saint-Venant.

2. To what extent is it valid to solve problems with Saint-Venant boundary conditions by means of Equations (0.2).

The question 1 is first clarified from a qualicative point of view (Section 1), and then a quantitative evaluation (Section 4) is given. The second question is clarified by means of an analysis of the solutions of Equation (0.2) and certain other equations, and a comparison of these solutions with exact solutions of the equations of dynamic elasticity for certain particular transient problems. Such comparisons were also carried out earlier [4, 8 and 9]; however they either compared solutions of stationary problems (*), or exact solutions were represented in the form of an expansion of stationary solutions (modes). Equation (0.2) also gives two such modes, whereas the exact solution leads to an infinite number of modes. To put all of these together so as to obtain a transient or stationary process with a specified form of disturbance is not possible. It would seem that if the higher modes (absent in (0.2)) are essential in the solution, then results given by (0.2) are not reliable. As will be shown subsequently, this is not always the case. The inadequacy of an expansion in modes forces one to resort to other methods of solution.

1. On the peculiarities of propagation of non-self-equilibrating and self-equilibrating end loads. We examine a semi-infinite beam to which at time $t = 0$ is applied a certain abrupt self-equilibrating load, which thereafter remains unchanged.

This problem reduces to the solution of Equation

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u} - \rho\mathbf{u}^{\cdot\cdot} = 0 \tag{1.1}$$

with the initial conditions

$$\mathbf{u} = 0, \quad \mathbf{u}^{\cdot} = 0 \quad \text{for } t = 0 \tag{1.2}$$

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*) It should be mentioned that the justification [5, 8 and 9] or refinement [7 and 10] of the equations by comparison of phase velocities is not always valid. What occurs is that waves corresponding to the lower branches of the dispersion curves for a beam as a three-dimensional body becomes surface waves for high frequencies, which cannot occur in models described by a "beam theory". Hence, the corresponding curves of phase velocities of three-dimensional and one-dimensional theories for high frequencies possess different forms of deformation and their coincidence is not to be taken as evidence for the use of the one-dimensional theory.

and the boundary conditions

$$\sigma_{xx} = f_x\delta_0(t), \qquad \sigma_{xy} = f_y\dot{\delta}_0(t), \qquad \sigma_{xz} = f_z\delta_0(t) \quad \text{for } x = 0 \qquad (1.3)$$
$$\mathbf{u} = 0 \quad \text{for } x = \infty$$

Here dots denote differentiation with respect to time, $\delta_0(t)$ is the Heaviside function, and the functions

$$f_x = f_x(y, z), \quad f_y = f_y(y, z), \quad f_z = f_z(y, z)$$

satisfy Equations

$$\int f_x ds = 0, \quad \int f_y ds = 0, \quad \int f_z ds = 0, \quad \int y f_x ds = 0$$
$$\int z f_x ds = 0, \qquad \int (z f_y - y f_z)\, ds = 0 \qquad (1.4)$$

We look for a solution in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}^* \qquad (1.5)$$

Here $\mathbf{u}_0(x, y, z)$ is the solution of the static problem for the same beam with the end loading

$$\sigma_{xx} = f_x, \qquad \sigma_{xy} = \dot{f}_y, \qquad \sigma_{xz} = f_z$$

Substituting (1.5) in (1.1) to (1.3) we come to the conclusion that the unknown vector $\mathbf{u}^*$ should satisfy Equation (1.1) for the initial conditions

$$\mathbf{u}^* = -\mathbf{u}_0(x, y, z), \qquad \mathbf{u}^{*\cdot} = 0 \qquad (t = 0) \qquad (1.6)$$

and the stresses corresponding to the displacements $\mathbf{u}^*$ for $t > 0$, should be zero on the side surfaces of the beam as well as on its end.

Hence the original transient problem has been partitioned into static and dynamic problems. In the latter, the motion of points of the beam are excited by initial displacements taken from the static problem. But by virtue of Saint-Venant's principle, the displacements $\mathbf{u}_0$ are of local character, being already practically damped out for x of the order of the cross-sectional dimension of the beam. It is physically obvious that the motion $\mathbf{u}^*$, excited by such a local disturbance, leads to the propagation of a packet of waves along the beam. The width of this packet will at first be close to the width of the zone of initial disturbance and subsequently will increase because of dispersion.

At an arbitrary instant of time the following equation should be satisfied

$$V_0 = V_* + T_* \qquad (1.7)$$

Here V_0 is the potential energy of the initial disturbance, and V_*, T_* are the potential and kinetic energies of points of the beam in the dynamic problem.

Turning now to the original problem, and taking into account that its solution is the sum of solutions of static and dynamic problems, we arrive at the conclusion that in this case the motion of points of the beam reduces to the propagation along the beam of a narrow wave packet. Near the beam end $x = 0$ the stress field corresponding to the static solution is very rapidly established (after the time t^* – the time of passage of the deformation wave over a distance of the order of the cross section of the beam).

For $t \to \infty$ (in practice for $t \gg t^*$) the sum of the kinetic and potential energies of points of the beam approach $2V_0$, of which one half remains in a small neighborhood of the cross section $x = 0$ and one half propagates along the beam.

A fundamentally different picture is obtained if one assumes that the end loading is non-self-equilibrating (i.e. even if one of the equalities (1.4) is not fulfilled). Then the above partition of the problem into two parts turns out not to be possible since for $t \to \infty$ the displacements $|\mathbf{u}| \to \infty$. Physically, this means that upon the sudden application of a non-self-equilibrating end load to a semi-infinite beam the corresponding displacements continuously increase with time and the sum of the potential and kinetic energies of points of the beam approach infinity for $t \to \infty$.

It is clear that in this case the disturbance fronts propagate with the velocities of dilatational or shear waves; however, the disturbance zone does not have the character of a narrow wave packet but spans a continuously widening region starting from the loaded end.

We now assume that both self-equilibrating and non-self-equilibrating loads are suddenly and simultaneously applied to the end, whereby the maximum values of the stresses for both loads are of the same order of magnitide. In view of what has been indicated above, the ratio $(V_1 + T_1)/(V_2 + T_2)$ (where V_1, T_1 are, respectively, the potential and kinetic energies of the beam excited by the self-equilibrating load and V_2, T_2 are excited by the non-self-equilibrating load) continuously decreases with time, approaching zero for $t \to \infty$.

For $t \gg t^*$ this ratio must become extremely small, whereby the corrections applied to the stresses because of the self-equilibrating load , can manifest themselves only in the immediate neighborhood of the loaded end and near the disturbance front and do not influence the displacements of the axis of the beam.

Hence, one may assert that upon the simultaneous application of both self-equilibrating and non-self-equilibrating loads the former may be neglected on the same basis as is done in static problems, i.e. in view of the essential local nature of the correction.

For the same reason Saint-Venant's principle (in the sense indicated above) is valid in the investigation of an important class of problems. The above derivation, however, does not extend to periodic loads. The latter may exert an influence of the stress and displacement fields over the entire length of the beam (independent of the type of loading). Therefore, it is not, in general, possible to neglect self-equilibrating end loads in comparison with non-self-equilibrating end loads in the latter case.

As a specimen problem for a theoretical investigation of the questions that have been posed above, we consider the plane deformation of an infinitely wide plate, without limiting ourselves to the bending problem. We shall give equations for the case of displacement that are symmetrically distributed with respect to the center of the plate as well. The latter case is of interest because it leads to the same questions as in the bending problem; however, it is essentially more straightforward from the mathematical point of view. For the construction of approximate equations we use the method of representing the displacements and stresses in a series of Legendre polynomials [11]. In dynamical problems this course is more logical than the traditional series representation in the distance from a point to the middle z-surface. In the first place, by using a Fourier series instead of a power series we can also include, almost without any additional restrictions, solutions with discontinuities of the first type (i.e. we may apply the theory to wave propagation problems). In the second place, by expanding the stresses in a series of Legendre polynomials, we may separate the self-equilibrating part of the stress field in a pure form from the non-self-equilibrating part over a cross section of the plate. This is important if one has in view an investigation of the possibility of using Saint-Vaenant's principle in dynamical problems.

2. An exact formulation of the problem; derivation of approximate equations from the exact equations. We consider the equations of motion of an elastic body, written for the case of plane deformation

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \tau^2} + C_{21}{}^2 \frac{\partial^2 u}{\partial z^2} + (1 - C_{21}{}^2) \frac{\partial^2 w}{\partial x\, \partial z} &= -q_x \\ \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial \tau^2} + C_{21}{}^2 \frac{\partial^2 w}{\partial x^2} + (1 - C_{21}{}^2) \frac{\partial^2 u}{\partial x\, \partial z} &= -q_z\end{aligned} \qquad (C_{21} = c_2/c_1) \qquad (2.1)$$

All of the coordinates and displacements in (2.1) are nondimensional, which is achieved by dividing the dimensional coordinates and displacements by the half-thickness h of the plate. The time τ in (2.1) is likewise nondimensional, whereby

$$\tau = (t / h)\, c_1, \qquad c_1 = \sqrt{(\lambda + 2\mu) / \rho}, \qquad c_2 = \sqrt{\mu / \rho} \tag{2.2}$$

Here c_1 is the velocity of dilatational waves, c_2 is the velocity of shear waves, (a unit interval of time thus corresponds to the time of propagation of a dilatational wave over a distance equal to the half-thickness of the plate), and q_x, q_z are nondimensional body forces obtained by dividing the corresponding components of the body forces by $(\lambda + 2\mu) / h$.

Multiplying each of Equations (2.1) by Legendre polynomial $P_n(z)$ (n = 0, 1, 2, ...) and integrating over the thickness of the plate, taking into account the absence of stresses on the planes $z = \pm 1$, we obtain for

$$u_n(x, \tau) = \int_{-1}^{1} uP_n(z)\, dz, \qquad w_n(x, \tau) = \int_{-1}^{1} wP_n(z)\, dz \tag{2.3}$$

two independent infinite systems of partial differential equations. In these equations, which will occur below, we use the notation

$$q_{xn}(x, \tau) = \int_{-1}^{1} q_x P_n(z)\, dz, \qquad q_{zn}(x, \tau) = \int_{-1}^{1} q_z P_n(z)\, dz \tag{2.4}$$

1. The system describing the longitudinal deformation (u is even, and w is an odd function of z)

$$\begin{gathered}
u_0'' - u_0^{\cdot\cdot} + 2\,(1 - 2C_{21}^2)\, w'(1) = -\, q_{x0} \\
u_2'' - u_2^{\cdot\cdot} + 3C_{21}^2 u_0 - 3\,(1 - C_{21}^2)\, w_1' - 6C_{21}^2\, u(1) + \\
+ 2\,(1 - 2C_{21}^2)\, w'(1) = -\, q_{x2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
C_{21}^2 w_1'' - w_1^{\cdot\cdot} - (1 - C_{21}^2)\, u_0' - 2w(1) + 2C_{21}^2 u'(1) = -\, q_{z1} \\
C_{21}^2 w_3'' - w_3^{\cdot\cdot} + 15w_1 - (1 - C_{21}^2)\, u_0' - 5\,(1 - C_{21}^2)\, u_2' - \\
- 12w(1) + 2C_{21}^2 u'(1) = -\, q_{z3} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{2.5}$$

Here $u(1)$, $w(1)$ are the values of u, w for $z = 1$

$$u(1) = \sum_{n=0,2,\ldots} \left(n + \tfrac{1}{2}\right) u_n, \qquad w(1) = \sum_{n=1,3,\ldots} \left(n + \tfrac{1}{2}\right) w_n \tag{2.6}$$

2. The system describing the flexural deformations (u is an odd and w is an even function of z)

$$\begin{gathered}
u_1'' - u_1^{\cdot\cdot} - (1 - C_{21}^2)\, w_0' - 2C_{21}^2\, u(1) + 2\,(1 - 2C_{21}^2)\, w'(1) = -\, q_{x1} \\
u_3'' - u_3^{\cdot\cdot} - 15C_{21}^2\, u_1 - (1 - C_{21}^2)\, w_0' - 5\,(1 - C_{21}^2)\, w_2' - 12C_{21}^2 u(1) + \\
+ 2\,(1 - 2C_{21}^2)\, w'(1) = -\, q_{x3} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{2.7}$$

$$C_{21}^2 w_0'' - w_0^{\cdot\cdot} + 2C_{21}^2 u'(1) = -q_{z0} \qquad \text{(2.7) cont.}$$
$$C_{21}^2 w_2'' - w_2^{\cdot\cdot} + 3w_0 - 3(1 - C_{21}^2) u_1' - 6w(1) + 2C_{21}^2 u'(1) = -q_{z2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Here

$$u(1) = \sum_{n=1,3,\ldots} \left(n + \tfrac{1}{2}\right) u_n, \qquad w(1) = \sum_{n=0,2,\ldots} \left(n + \tfrac{1}{2}\right) w_n \qquad (2.8)$$

By retaining in these systems a finite number of equations, one may obtain approximate variants of the theory. In this procedure, if one retains terms up to u_{2n}, w_{2-1} (in the first case) or terms up to u_{2n+1}, w_{2n} (in the second case) in the expansions of the displacements in Legendre polynomials, then it is necessary to delete terms in the right-hand sides corresponding to the terms neglected in the series. In view of the linearity of the problem, the left-hand sides of the differential equations for all u_k, w_k corresponding to an arbitrary approximation will be the same. We give some of the operators of the left-hand sides of these equations for some first approximations.

Retained quantities	Operator	
	Longitudinal deformations	
u_0	L_1	(2.9)
u_0, w_1	$L_1L_2 + 3C_{32}^2 [L_3]$	(2.10)
	Flexural deformations	
w_0, u_1	$L_1L_2 + 3\dfrac{\partial^2}{\partial\tau^2}$	(2.11)
$w_0,\ u_1,\ w_2$	$L_1L_2^2 + 15C_{32}^2\left[L_4L_5 + 3C_{13}^2\dfrac{\partial^2}{\partial\tau^2}\right]$	(2.12)
$w_0,\ u_1,\ w_2,\ u_3$	$L_1^2L_2^2 + 15\ C_{32}^2L_6L_7L_8 + 525\ C_{31}^2\left(L_9L_{10} + 3C_{13}^2\dfrac{\partial^2}{\partial\tau^2}\right)$	(2.13)

Here

$$L_i = \frac{c_1^2}{c_i^2}\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2} \qquad (i = 1, 2, \ldots, 10), \qquad C_{ij} = \frac{c_i}{c_j} \qquad (2.14)$$

The expression for the propagation velocity of longitudinal displacements c_3 corresponding to the state of plane stress has the form

$$c_3 = \left(\frac{4\mu(\lambda + \mu)}{\rho(\lambda + 2\mu)}\right)^{1/2} = \left(\frac{E}{(1 - \nu^2)\rho}\right)^{1/2} \qquad (2.15)$$

The parameter c_i in (2.14), having the dimension of velocity, takes on the following values for $\lambda = 1.4\mu$ ($\nu = 0.292$) :

$i = 2$	3	4	5	6	7	8	9	10
$C_{i1}^2 = 0.294$	0.830	0.805	0.292	0.99	0.292	0.347	0.738	0.246

From these results one may observe the following.

1) For an arbitrary degree of accuracy the derived approximate equations

give propagation velocities of discontinuities which are equal to c_1 and c_2. These coincide with the velocities of dilatational and shear waves (2.2).

The latter is a consequence of the fact that terms with higher derivatives in each operator are always products of power of two homogeneous wave operators L_1 and L_2, to which corresond just these velocities.

2) The equations with the operators (2.9) and (2.11) (i.e. the first approximations for problems of longitudinal and flexural oscillations of a plate), in which are considered only non-self-equilibrating terms in the series for the stresses, coincide with the standard equations for longitudinal oscillations of a plate in plane stress and with the Timoshenko equations. However, the values of the coefficients in the latter equations do not coincide with (2.9) and (2.11).

Indeed, in the adopted notation the equation of longitudinal oscillations has the following form

$$\frac{\partial^2 u_0}{\partial x^2} - C_{12}^2 \frac{\partial^2 u_0}{\partial \tau^2} = 0 \tag{2.16}$$

While the Timoshenko equation is written in the form

$$\left[\left(\frac{\partial^2}{\partial x^2} - \alpha_1^2 \frac{\partial^2}{\partial \tau^2}\right)\left(\frac{\partial^2}{\partial x^2} - \alpha_2^2 \frac{\partial^2}{\partial \tau^2}\right) + 3C_{12}^2 \frac{\partial^2}{\partial \tau^2}\right] w_0 = 0 \tag{2.17}$$

where (adapted to plane deformation of a plate)

$\alpha_1 = 1.10,\quad \alpha_2 = 2.02$ (after Timoshenko [13])
$\alpha_1 = 1.10,\quad \alpha_2 = 2.26$ (after Ufliand [1])
$\alpha_1 = 1.26,\quad \alpha_2 = 2.05$ (after Selezov)

The latter values of α_1 and α_2 were obtained by expanding the solutions in a power series in z and discarding the infinite number of terms with derivatives higher than fourth order in the equations that are obtained.

It is known that Equations (2.16) and (2.17) cover a number of problems in which the displacements and stresses are sufficiently smooth (slowly changing) functions of x and τ. Hence it follows that equations corresponding to the operators (2.9) and (2.11) which correctly reveal the character of the most rapidly changing parts of the solutions, must give the essential error in determining slowly changing displacements and stresses.

3) The latter insufficiency is eliminated if one turns to the second approximations to which operators (2.10) and (2.12) correspond.

In fact, the terms in square brackets of the operator (2.10) are identical to (2.16), while the terms of the operator (2.12) are close to the coefficients of (2.17). At the same time the asymptotics of the operator (2.12) for slow processes

$$\frac{\partial^4}{\partial x^4} + 3C_{12}^2 \frac{\partial^2}{\partial \tau^2}$$

coincide with the asymptotics of the operator (2.17).

Therefore, the equations of the second approximation describing the asymptotic behavior of the most rapidly changing processes also give valid solutions for slow processes. In the light of the above, the equation of longitudinal oscillations (2.16) and the equation of flexural oscillations (2.17) may be treated (from the standpoint of applying general methods) as second approximations of (2.10) and (2.12), in which are disregarded terms of higher derivatives (fourth order in (2.10) and sixth order in (2.12)).

Less formal interpretations of Equations (2.16) and (2.17) may also be given.

In formulating the operators (2.9) and (2.11), all self-equilibrating components were disregarded. Without raising the order of the operators, one may proceed differently: one may take into account displacements corresponding to the first of the self-equilibrating components of body forces, assuming an asymptotic dependence (for slow processes). Thus, for $|w_1^{\cdot\cdot}| \ll 3|w_1|$, $|w_1''| \ll 3C_{12}^2|w_1|$ from the third of Equations (2.5) we obtain

$$w_1 = -{}^1/_3\,(1 - 2C_{21}^2)\, u_0'$$

and substituting into the first equation, we find

$$L_3(u_0) = C_{13}^2 q_{x0} \tag{2.18}$$

which is the wave equation for plane deformation of a plate.

Neglecting derivatives of w_2 and u_3 in the second and third equations of the system (2.7), we obtain (*) Equations

$$\begin{aligned} w_0'' - {}^6/_5 C_{12}^2 w_0^{\cdot\cdot} + 3u_1' &= -{}^6/_5 C_{12}^2 q_{z0} \\ u_1'' - C_{13}^2 u_1^{\cdot\cdot} - {}^5/_6 C_{23}^2 (3u_1 + w_0') &= -C_{13}^2 q_{x1} \end{aligned} \tag{2.19}$$

with the basic operator (2.17) identical to the system of equations of Timoshenko [12].

Thus the "engineering equations" – the equation of longitudinal oscillations and the Timoshenko equations, are the consequence of the theory of elasticity if one considers processes in which displacements corresponding to the first of the self-equilibrating components of the body forces change sufficiently slowly while the remaining components may be neglected. By the indicated assumptions, a simplification of the equations is attained, but one loses accuracy in the determination of the velocity of propagation of discontinuities and in the description of the stress and displacement fields in their neighborhoods.

4) Without a special investigation, it is impossible to say how much this loss is essential for evaluation of the practical significance of the equation of longitudinal oscillations and the Timoshenko equation. As will become clear in the sequel, in problems of propagation of deformations in plates and beams the interest is focussed not only on the actual front but also on the quasi-front, on which the stresses, although not suffering discontinuities, have essentially larger gradients. The energy of the wave packet immediately following the actual front is relatively small for sufficiently large distances from the source of the disturbance $(x \gg 1)$. The overwhelming part of the energy follows the quasi-front. This significantly decreases the interest in describing the motion in the neighborhood of the front and forces one to focus attention on the region where the larger part of the energy of motion is concentrated.

The latter should be kept in mind when one considers the feasibility of approximate equations for the dynamics of plates and beams. Moreover, considering that a correct estimate must show a preferential distribution of enrgy, it is impossible to unrestrictedly reject even the Bernoulli-Euler equation (0.1) as an apparatus for the study of the propagation of bending deformations along the beam on the basis that in this equation one assumes $\alpha_1 = \alpha_2 = 0$, i.e. that the velocity of wave propagation is assumed infinite. In the subsequent Sections we give a number of examples which illustrate this and which throw light on the degree of accuracy and the region of applicability of various approximate variants of the equations of the dynamics of beams and plates. In passing, we give some quantitative results relative to the propagation of self-equilibrating disturbances along a beam (plate).

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*) Sometimes [5] it is asserted that the Timoshenko equations are based on the assumption of plane cross sections. In the derivation of the Timoshenko equations that has been given here, the assumption of plane cross sections was not used $(u_3 \not\equiv 0)$.

3. Transition processes in the problem of longitudinal deformations of plates. As a subject for investigation and comparison of exact and approximate equations of motion of beams, we consider a transient problem which is from a mathematical point of view one of the simplest. This is the excitation of plane motion of an infinite plate by a concentrated body force on the plate yz changing according to the law

$$q_x = Q_x(z)\,\delta_1(x)\,\delta_0(\tau), \qquad q_z = Q_z(z)\,\delta_1(x)\,\delta_0(\tau) \tag{3.1}$$

Here δ_1 is the Dirac function, δ_0 is the Heaviside function, and initially ($\tau = 0$) the plate is at rest $(u = w = u^{\bullet} = w^{\bullet} = 0)$.

This somewhat artificial problem allows one to use not only the Laplace transform but the Fourier transform as well, which is its advantage compared to more realistic problems wherein the plate is semi-infinite and its motion is excited by boundary loads varying in an analogous way.

We start with the case where Q_x is an even and Q_z is an odd function of z, i.e. with nonflexural deformations of the plate. We construct a solution proceeding from the equations of the second approximation to which corresponds the operator (2.10). This operator, in addition to the non-self-equilibrating stresses over the cross section, also takes into account the sef-equilibrating stresses σ_{xz} which vary linearly. In this situation Equations (2.5) take on the form

$$u_0'' - u_0^{\bullet\bullet} + 3\,(1 - 2C_{21}^2)\,w_1' = -\,Q_{x0}\delta_1(x)\,\delta_0(\tau) \tag{3.2}$$

$$-\,(1 - 2C_{21}^2)\,u_0' + C_{21}^2\,w_1'' - w_1^{\bullet\bullet} - 3w_1 = -\,Q_{z1}\delta_1(x)\,\delta_0(\tau)$$

$$Q_{x0} = \int_{-1}^{1} Q_x(z)\,dz = 2Q, \quad Q_{z1} = \int_{-1}^{1} zQ_z(z)\,dz = 2R \tag{3.3}$$

After applying to (3.2) the Fourier transform (^{F}) in x and the Laplace transform (^{L}) in τ we obtain

$$(q^2 + p^2)\,u_0^{LF} + 3\,(1 - 2C_{21}^2)\,iqw_1^{LF} = 2Q\,/\,p \tag{3.4}$$

$$-\,(1 - 2C_{21}^2)\,iqu_0^{LF} + (C_{21}^2q^2 + p^2 + 3)\,w_1^{LF} = 2R\,/\,p$$

Therefore

$$u_0^{LF} = A^{-1}(p^2, q^2)\,[(C_{21}^2q^2 + p^2 + 3)\,2Q\,/\,p - 3iq\,(1 - 2C_{21}^2)\,2R\,/\,p]$$

$$w_1^{LF} = A^{-1}(p^2, q^2)\,[iq\,(1 - 2C_{21}^2)\,2Q\,/\,p + (q^2 + p^2)\,2R\,/\,p] \tag{3.5}$$

$$A(p^2, q^2) = p^4 + [(1 + C_{21}^2)\,q^2 + 3]\,p^2 + C_{21}^2q^4 + 3C_{31}^2q^2 \tag{3.6}$$

Since $\sigma_{xx0}^{LF} = -\,iqu_0^{LF} + 3\,(1 - 2C_{21}^2)\,w_1^{LF}$ one may write

$$\sigma_{xx0}^{LF} = \left[\frac{-iq}{q^2 + p^2} + 3\left(1 - 2C_{21}^2\right)^2 \frac{iqp^{21}}{(q^2 + p^2)\,A(p^2, q^2)}\right]\frac{2}{p}\,Q +$$

$$+\,3\left(1 - 2C_{21}^2\right)\frac{p^2}{A(p^2, q^2)}\,\frac{2}{p}\,R \tag{3.7}$$

Hence

$$\sigma_{xx0} = \frac{Q}{\pi}\int_{-\infty}^{\infty}\left[-\frac{i}{q}+3\left(1-2C_{21}^2\right)^2\frac{iq}{p_1^2-p_2^2}\left(\frac{\cos p_2\tau}{q^2-p_2^2}-\frac{\cos p_1\tau}{q^2-p_1^2}\right)\right]\times$$

$$\times e^{-iqx}\,dq+\frac{R}{\pi}\int_{-\infty}^{\infty}3\left(1-2C_{21}^2\right)\frac{\cos p_2\tau-\cos p_1\tau}{p_1^2-p_2^2}e^{-iqx}\,dq \qquad (3.8)$$

Here $(-p_1^2)$, $(-p_2^2)$ are the roots of Equation $A(p^2, q^2) = 0$

$$p_{1,2}^2 = {}^1/_2\,[(1+C_{21}^2)q^2+3] \pm \sqrt{{}^1/_4\,[(1+C_{21}^2)q^2+3]^2 - C_{21}^2q^4 - 3C_{31}^2q^2} \qquad (3.9)$$

Graphs of the functions of $p_1(q)$, $p_2(q)$, $\partial p_1/\partial q$, $\partial p_2/\partial q$ are shown in Fig.1.

If $\tau \to \infty$, the basic contribution to the value of the first integral (3.8) will be given by the integration of the first term and likewise the integration in the neighborhoods of the zeros of the denominators of the terms in parentheses (i.e. in the neighborhood of $q = 0$ and $q = \infty$).

In this case one may apply the expansion

$$p_1^2 = 3 + 0.463q^2 + \ldots, \qquad p_1 = \sqrt{3} + 0.1337q^2 + \ldots \quad \text{for } q \to 0 \qquad (3.10)$$

$$p_2^2 = 0.830q^2 - 0.0305q^4\ldots, \qquad p_2 = 0.911q - 0.0167q^3 + \ldots$$

$$p_1^2 = q^2 + 0.722 + \ldots, \qquad p_1 = q + 0.361q^{-1} + \ldots \quad \text{for } q \to \infty \qquad (3.11)$$

$$p_2^2 = 0.294q^2 + 2.278 + \ldots, \qquad p_2 = 0.541q + 2.06q^{-1} + \ldots$$

Restricting ourselves for the time being to the case of non-self-equilibrating disturbances, i.e. setting $R = 0$, we have

$$\frac{\sigma_{xx0}}{Q} = -1 + 3 - 2C_{21}^2)^2\,\frac{1}{\pi}\int_0^{\infty}\frac{q}{p_1^2-p_2^2}\sum_{i=1}^{2}\frac{\sin(qx+p_i\tau)+\sin(qx-p_i\tau)}{q^2-p_i^2}(-1)^i\,dq \qquad (3.12)$$

whereby on the basis of the above considerations for $\tau \to \infty$

$$\frac{\sigma_{xx0}}{Q} \sim -1 + \frac{1}{\pi}\int_0^{\infty}\sum_{i=1}^{2}\left[\sin\,(qx + (-1)^i m) + \sin\,(qx + (-1)^i n)\right]\frac{dq}{q}$$

$$(m = (0.911q - 0.0167q^3)\,\tau, \quad n = (q + 0.361q^{-1})\,\tau) \qquad (3.13)$$

From the approximate solution that has been obtained it follows that:

a) For $x = c\tau$ $(c < C_{31} = 0.911)$ the basic contribution is given by the first term of (3.13) (in the second term two components mutually cancel), i.e. in this range

$$\sigma_{xx0} \sim -Q \qquad (3.14)$$

b) For $x = 0.911\tau + \varepsilon$, the first and second terms of (3.13) give the basic contribution, whereby

$$\sigma_{xx0} \sim -Q\left(\frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\sin\left(q\varepsilon + 0.0167q^3\tau\right)\frac{dq}{q}\right) \qquad (3.15)$$

If one lets $\alpha = (0.0167\tau)^{-1/3}\varepsilon$, then Formula (3.15) may be rewritten in the form

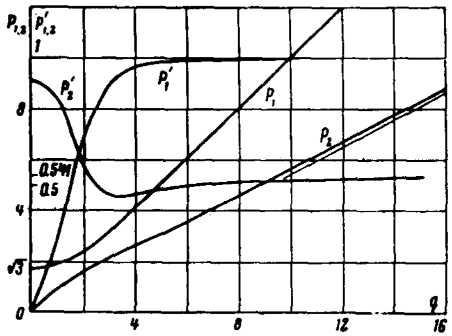

Fig. 1

$$\sigma_{xx0} \sim -Q\left(\frac{1}{3} - \int_0^{\alpha} Ai(\varphi)\, d\varphi\right)$$

$$\left(Ai(\varphi) = \frac{1}{\pi}\int_0^{\infty} \cos(t^3 + \varphi t)\, dt\right) \qquad (3.16)$$

Here $Ai(\varphi)$ is the Airy function [13] whose graphs may be found in [14].

c) For $x = \tau - \varepsilon$ ($\varepsilon > 0$) the last term of (3.13) gives the basic contribution (the first two terms mutually cancel) and

$$\sigma_{xx0} \sim -QJ_0\,(1.20\sqrt{\varepsilon\tau}) \qquad (3.17)$$

Therefore, Equations (3.1) give the following picture of propagation of deformation upon the application of a sudden force $Q_x = 2Q\delta_1(x)\,\delta_0(\tau)$.

1) Disturbances with jumps in the stresses Q propagate with the velocity of dilatational waves (c_1) . At a sufficient distance from the plane of the disturbance $x = 0$ the wave packet following behind degenerates to a narrow peak-signal which carries a small part of the energy (approaching zero for $\tau \to \infty$)

2) A quasi-front propagates with the velocity c_3 . In the region of the quasi-front

$$Q^{-1}\frac{\partial\sigma_{xx0}}{\partial\tau} = Ai(0)\,\frac{\partial\alpha}{\partial\tau} = -\frac{\Gamma(1/3)\cos 1/6\pi}{3\pi}\,0.911\,(0.0167\tau)^{-1/3} = -0.88\tau^{-1/3} \quad (3.18)$$

3) In the interval between $x = 0$ and the quasi-front the stresses are close to the constant value $-Q$.

We compare the results that have been obtained with the exact solution of the same problem arising immediately from (2.1) and with a cruder approximate solution based on the equation of longitudinal oscillations (2.18).

In this case the equation of longitudinal oscillations leads to the loss of an actual front and of the narrow zone of disturbance in the immediate neighborhood of it, whereas the quasi-front turns into an actual front with a jump in the stresses Q . In the region between the front and $x = 0$ the obtained stresses are constant : $\sigma_{xx0} = -Q$. As far as the exact solution of the problem is concerned, applying to (2.1) the Fourier and Laplace integral transforms we come to the following formula for the Laplace presentation:

$$Q^{-1}\sigma_{xx0}^{L} = -\frac{1}{p} + \frac{2}{\pi}\,(C_{12}{}^2 - 2)^2\, p^3 \int_0^{\infty} \frac{q \sin qx\, dq}{(q^2 + p^2)\, B(p^2, q^2)} \qquad (3.19)$$

$$B(p^2, q^2) = (q^2 + n_2^2)^2 n_1 \coth n_1 - 4q^2 n_2 n_1^2 \coth n_2$$
$$n_1 = \sqrt{q^2 + p^2}, \qquad n_2 = \sqrt{q^2 + C_{12}^2 p^2} \tag{3.20}$$

Having in view a solution for sufficiently large x and τ, we expand the hyperbolic functions entering in B into a power series and retain the first two terms.

Then we obtain for σ_{xx0} an expression which differs from (3.8) only in the values of the roots $(-p_1^2)$, $(-p_2^2)$, namely (3.21)

$$p_2^2 = 0.830q^2 - 0.0470q^4 + \ldots, \qquad p_1^2 = 3 + 0.170q^2 + \ldots \quad \text{for } q \to 0$$

This difference leads to a certain velocity, different from that in (3.18) (less by 13%), of the change of the stresses in the region of the quasi-front, namely,

$$\frac{\partial \sigma_{xx0}}{\partial \tau} = -0.77\tau^{-1/3} \quad (\text{for } x = C_{31}\tau) \tag{3.22}$$

In all of the rest, the picture which was obtained earlier on the basis of the second approximation (3.2) is confirmed. By different means the investigation in [14] comes to the same results (for the region $x = C_{31}\tau$), which are also qualitatively confirmed by experiments (*).

The above methods are not suitable for the investigation of the deformation in the initial period of motion (for relatively small τ). Here one may effectively use an expansion of the deformation into a Fourier series over an interval which is varied so as to completely cover the deformed part of the plate $(0 \leqslant |x| \leqslant \tau)$.

The advantages of this method are the possibility of applying a Fourier series to transient problems for unbounded regions. Also in transient problems for bounded regions convergence of the trigonometric series is improved in that interval of time in which the disturbance has not yet propagated over the entire region. Formally, the indicated solution is obtained from the solution corresponding to the constant interval $2l$ with a subsequent change of $2l$ into 2τ.

In the present problem, when the second of the approximations (3.2) is used, the Fourier series on the interval $2l = 2\tau$ is obtained from the Fourier integral (3.12) by an argument which is the reverse of that which is usually applied to obtain the Fourier integral from the Fourier series, i.e. by replacing q by $n\pi/\tau$ (n is the number of the term of the series) and the integral by a sum and by multiplying the zero order term of the series by π/τ and the remaining terms by $2\pi/\tau$. In this case we obtain

$$\frac{\sigma_{xx0}}{Q} = -1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} + \frac{\pi^2}{\tau^2}3\left(1 - 2C_{12}^2\right)^2 \frac{n}{p_1^2 - p_2^2}\sum_{i=1}^{2}(-1)^i \frac{\cos p_i\tau}{q^2 - p_i^2}\right]\sin qx \quad \text{for } 0 < x < \tau$$
$$Q_{xx0}^{-1} = 0 \qquad \text{for } x > \tau \tag{3.23}$$

Here p_1, p_2 are defined, as above in (3.9), by the substitution $q = n\pi/\tau$

The results of the calculation according to Formula (3.23), illustrating the evolution and establishment of the stress wave in the plate, are shown in Fig.2. The calculation is carried out for values of τ from 0.5 to 10. For subsequent values one may use the estimates which were given at the

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*) In [14] there is an error in Formula (41):; in the first of the square brackets the numerator is equal to 6 and not 2.

beginning of the Section. The dotted lines in Fig.2 correspond to the elementary theory.

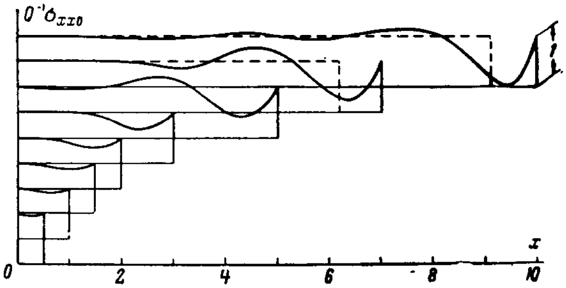

Fig. 2

4. Investigation of the Saint-Venant effect in the problem of longitudinal oscillations of a plate. In the previous Section the case $Q \neq 0$, $R = 0$ was investigated. The converse case $R \neq 0$, $Q = 0$ also is of interest inasmuch as in this case the body forces q_z are self-equilibrating and, therefore, the propagation of the deformation should correspond to the Saint-Venant principle (i.e. in the form of a narrow wave packet continuously "melting" because of dispersion).

It is clear that in the case under consideration the displacement u_0 does not have a deciding influence on the propagation along the plate of self-equilibrating shear stresses.

Substituting into the second of Equations (3.4) the quantity $u^{LF} = 0$, we obtain

$$(C_{21}^2 q^2 + p^2 + 3)\, w_1^{LF} = 2R\,/\,p \tag{4.1}$$

Hence we have the simple value

$$w_1^{F\cdot} = 2R\,\frac{\sin\sqrt{C_{21}^2 q^2 + 3}\,\tau}{\sqrt{C_{21}^2 q^2 + 3}} \tag{4.2}$$

$$R^{-1} w_1 = \frac{c_1}{c_2} J_0\left(\sqrt{3\,(\tau^2 - C_{12}^2 x^2)}\right) \qquad (c_2\tau > c_1 x) \tag{4.3}$$

For large τ and small $(\tau - x)\tau^{-1}$

$$w_1' = -\,C_{12} w_1^{\boldsymbol{\cdot}} + O\,(\tau^{-1}) \tag{4.4}$$
$$\sigma_{xz1} = C_{21} w_1' \approx -\,R J_0\left(\sqrt{3\,(\tau^2 - C_{12}^2 x^2)}\right) \qquad (c_2\tau > c_1 x)$$

In the region where the external disturbances are applied

$$\sigma_{xz1} \approx -R\exp\left(-\sqrt{3}\,C_{12} x\right) \tag{4.5}$$

The propagation process of stresses that are self-equilibrating over the cross section (σ_{xz1}) in the initial period of motion, may be studied more exactly (without disregarding u_0) by expanding the stresses into a Fourier series over a time-varying interval.

Proceeding from (3.5) and the connection of the stresses to the displacements we find

$$R^{-1}\sigma_{xz1} = -\delta_0(C_{21}\tau - x) + \frac{2}{\pi}C_{21}^2\sum_{n=1}^{\infty}\left\{\frac{1}{n}\frac{q^2}{p_1^2-p_2^2}\left[\cos p_1\tau - \cos p_2\tau + \right.\right.$$
$$\left.\left. + q^2\sum_{i=1}^{2}(-1)^{i+1}\frac{1-\cos p_i\tau}{p_i^2}\right] + C_{12}^2\frac{1}{n}\left(1-\cos n\pi\ \ C_{21}\right)\right\}\sin qx \quad (4.6)$$

where $p_{1,2}$ are defined in the same way as in (3.23).

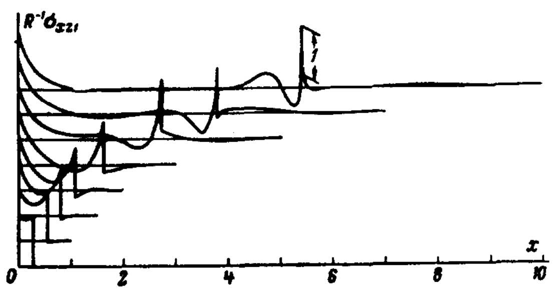

Fig. 3

The results of calculations according to Formula (4.6) for various values $0.5 \leqslant \tau \leqslant 10$ are shown in Fig.3. From this it follows that starting with $\tau = 5$ the deformation already converges to a wave packet of width of the order of unity which propagates with the velocity of a shear wave c_2 and to the establishment in the neighborhood of $x = 0$ of a stress field which is near the static field corresponding to the applied load. It is clear (as a consequence of the arguments given in Section 1 and the evaluations given above) that with increasing τ this picture is retained. Hence the continuation of the calculations past $\tau > 10$ does not give anything new.

In the case of periodic disturbances, a consideration of the connection between the displacements and self-equilibrating and non-self-equilibrating stresses is essential in a study of the Saint-Venant effect.

If such a connection is ignored (the study of an unbounded medium), one is led to the following conclusion [16]: the Saint-Venant effect ("boundary layer") takes place for frequencies below a certain critical frequency (depending on the frequency of the form of excitation) and is absent for higher frequencies. In the case of a plate, a different conclusion is reached.

Changing in (3.2) the value of $\delta_0(\tau)$ to $e^{i\omega\tau}$, and proceeding from the connection between the stresses and deformations, we obtain for $x > 0$

$$R^{-1}\sigma_{xz1} = -e^{i\omega\tau}(q_1^2 - q_2^2)^{-1}\left[(q_1^2+\omega^2)e^{-q_1x} - (q_2^2+\omega^2)e^{-q_2x}\right] \quad (4.7)$$

$$q_{1,2}^2 = {}^1/_2\left[-(1+C_{12}^2)\omega^2 + 3C_{32}^2\right] \pm$$
$$\pm\sqrt{{}^1/_4(C_{12}^2-1)^2\omega^4 - 3(C_{12}^2-2C_{21}^2)\omega^2 + 36(1-C_{21}^2)^2}$$
$$q_1^2 > 0,\ \ q_2^2 < 0 \ \text{ for } 0<\omega^2<3; \qquad q_1^2 < 0,\ \ q_2^2 < 0 \text{ for } \omega^2 > 3$$

The root q_2 is always imaginary and therefore the corresponding part of of the stresses σ_{xz1} does not damp with an increase in the coordinate value and the x-effect of Saint-Venant is absent. The effect becomes apparent for $\omega^2 < 3$ only for the part of the stresses corresponding to the root q_1.

Thus in stationary processes self-equilibrating loads continuously induce stress waves which propagate along the plate.

It should be mentioned that for both stationary and transient processes a self-equilibrating disturbance excites also non-self-equilibrating stresses.

As a consequence of this, the stress waves described above, propagating along the plate, carry only part of the energy emitted by the external disturbance.

However in the transient case the total energy leaving the region of the self-equilibrating disturbance remains equal to V_0 (see Section 1).

Above, we solved a particular problem that shed light on the peculiarity of propagation of self-equilibrating and non-self-equilibrating disturbances for longitudinal deformations of a plate. A comparison for $\tau \gg 1$ of the solution of Equations (3.2) with results from the theory of elasticity fundamentally confirm the validity of this solution. One cannot say that this conclusion is unexpected if one considers that for $\tau \gg 1$ the quasi-front smoothes out, as a consequence of which the higher components in the Legendre polynomial expansion of the displacements become unessential.

The acceptability of the "engineering" equations for describing dynamic deformations for small τ and suddenly applied loads is usually questioned. Below we carry out a comparison of solutions of (3.2) and the theory of elasticity for small τ.

5. The dynamic flexibility of a plate for small τ. We investigate the dynamic flexibility of a plate under the action of body forces

$$q_x = 2Q\delta_1(x)\,\delta_0(\tau)$$

We will first find a solution of the equations of elasticity.

The homogeneity of the characteristic polynomial corresponding to the equations of elasticity (2.1) (in (2.1) only second order derivatives enter) allows one to construct a solution in the form of separate waves. This is convenient to use [15 and 20] for small τ.

The solution is carried out by the application of Laplace and Fourier transforms. In this as a consequence of the homogeneity of the transform representing waves reflected from the free surfaces of the plate, it turns out to be possible to establish the formal identity of the inverse Fourier transform with the direct Laplace transform. As a result, the necessity of carrying out both the $(^L)$ and $(^F)$ inverse transformations is eliminated.

In the case of longitudinal deformations, we find by means of a Laplace transformation in τ and a Fourier transformation in x from Equation (2.1) that

$$\begin{gathered}(q^2+p^2)\,u^{LF} - C_{21}^2\frac{d^2u^{LF}}{dz^2} + \left(1-C_{21}^2\right)iq\,\frac{dw^{LF}}{dz} = \frac{2}{p}\,Q \\ \frac{d^2w^{LF}}{dz^2} - \left(C_{21}^2 q^2 + p^2\right)w^{LF} - \left(1-C_{21}^2\right)iq\,\frac{du^{LF}}{dz} = 0\end{gathered} \tag{5.1}$$

Considering the boundary conditions for $z = \pm 1$

$$\frac{dw^{LF}}{dz} + \frac{1}{2}\left(C_{21}^2 - 2\right)\left(-iqu^{LF} + \frac{dw^{LF}}{dz}\right) = 0, \quad -iqw^{LF} + \frac{du^{LF}}{dz} = 0 \tag{5.2}$$

The solution of the system (5.1) for $u_0^L(0, p)$ has the form

$$\frac{u_0^L(0,p)}{Q} = \frac{1}{p^2} + \left(C_{21}^2 - 2\right)\frac{2}{\pi}\,p\int_0^\infty \frac{q^2dq}{(q^2+p^2)\,B(p^2,q^2)} \tag{5.3}$$

where B is determined by Expression (3.20).

The hyperbolic functions which enter into B can be represented in the form

$$\coth n_{1,2} = 1 + 2e^{-2n_{1,2}} + 2e^{-4n_{1,2}} + \ldots \tag{5.4}$$

Then

$$B = a_1 - a_2 + 2a_1 (e^{-2n_1} + e^{-4n_1} + \ldots) - 2a_2 (e^{-2n_2} + e^{-4n_2} + \ldots) \tag{5.5}$$

$$\frac{u_0{}^L(0, p)}{Q} = \frac{1}{p^2} + \left(C_{21}{}^2 - 2\right)^2 \frac{2}{\pi} p \int_0^\infty \frac{q^2 dq}{n_1{}^2 (a_1 - a_2)} \left[1 - 2\frac{a_1 e^{-2n_1} - a_2 e^{-2n_2}}{a_1 - a_2} + \ldots\right]$$

$$(a_1 = (q^2 + n_2{}^2) n_1, \quad a_2 = 4q^2 n_1{}^2 n_2) \tag{5.6}$$

The terms in (5.6) containing exponential factors correspond to delayed inverses, so that for $\tau < 2$ they may all be ignored, for $\tau < 2C_{12}$ it is necessary to retain only the term with $\exp(-2n_1)$, and so forth.

Limiting ourselves to the time interval $0 < \tau < 2$, C_{12} we have

$$Q^{-1} u_0{}^L = p^{-2} + (C_{12}{}^2 - 2)^2 \ (C^L - D^L) 2/\pi \tag{5.7}$$

where

$$C^L = p\int_0^\infty \frac{q^2 dq}{n_1{}^3 [(q^2 + n_2{}^2)^2 - 4q^2 n_1 n_2]}, \qquad D^L = p\int_0^\infty \frac{2q^2 (q^2 + n_2{}^2)^2 e^{-2n_1} dq}{n_1{}^3 [(q^2 + n_2{}^2)^2 - 4q^2 n_1 n_2]^2} \tag{5.8}$$

We shall consider the Laplace transforms C^L and D^L only on the real axis p. Setting $q = ps$ in the integrals (5.8) we obtain

$$C^L = \alpha p^{-3}, \qquad C = {}^1/_2 \alpha \tau^2 \tag{5.9}$$

where

$$\alpha = \int_0^\infty \frac{s^2 ds}{(s^2 + 1)^{3/2} [(2s^2 + C_{12}{}^2)^2 - 4s^2 \sqrt{s^2 + 1} \sqrt{s^2 + C_{12}{}^2}]} = 0.0455$$

For the calculation of D we introduce the substitution

$$\sqrt{s^2 + 1} = 1 + {}^1/_2 \tau.$$

Then

$$D^L = \frac{e^{-2p}}{p^3} \int_0^\infty f(\tau) e^{-p\tau} d\tau$$

$$f(\tau) = \frac{\sqrt{{}^1/_4\tau^2 + \tau} \, ({}^1/_2\tau^2 + 2\tau + C_{12}{}^2)^2}{({}^1/_2\tau + 1)^2 [({}^1/_2\tau^2 + 2\tau + C_{12}{}^2)^2 - 4({}^1/_4\tau^2 + \tau)({}^1/_2\tau + 1) \sqrt{{}^1/_4\tau^2 + \tau + C_{12}{}^2}]} \tag{5.10}$$

The integral (5.10) may be looked upon as a Laplace transform of the function $f(\tau)$. Considering, in addition, the presence of the factor in front of the integral, we may write

$$D = \frac{1}{2} \int_2^{\tau - 2} f(\alpha - 2)(\tau - \alpha)^2 d\alpha \qquad \text{for } \tau > 2, \ D = 0 \ \text{ for } \tau < 2 \tag{5.11}$$

A graph of the velocity u_0 for a suddenly applied longitudinal load ($x = 0$) is given in Fig.4, where curve 1 is for the theory of elasticity, curve 2 for Equation (5.12), curve 3 for Equation (5.14), and curve 4 for Equation (3.2)

We next examine the nature of the dynamic flexibility according to models of an infinite plate which are described by simplified equations.

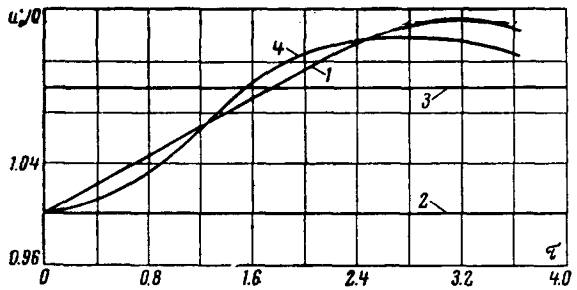

Fig. 4

1. The equation of longitudinal deformations considering only, u_0, that is, the basic operator L_1 (2.9)

$$u_0'' - u_0^{\cdot\cdot} = - q_{x0} = - 2Q\delta_1(x)\,\delta_0(\tau) \qquad (5.12)$$

the solution of this equation that is sought has the form

$$u_0^{\cdot}(0, \tau) = Q \quad (\tau > 0) \quad (5.13)$$

2. The equation of longitudinal deformations considering u_0, w_1 and the assumption of the smoothness of $w_1(x, \tau)$, usually used for a plate, (2.18)

$$C_{13}^2 u_0'' - u_0^{\cdot\cdot} = - 2Q\delta_1(x)\,\delta_0(\tau) \qquad (5.14)$$

its solution

$$u_0^{\cdot}(0, \tau) = C_{13}Q \qquad (\tau > 0) \qquad (5.15)$$

3. The equation of longitudinal deformations considering u_0, w_1, i.e. Equations (3.2).

Let $q_{x0} = 2Q\delta_1(x)\,\delta_0(\tau)$, $q_{z1} = 0$. then by means of the Laplace transform in τ one may obtain

$$u_0^L(0, p) = \frac{p^2 + m_1 m_2}{p^3 (m_1 + m_2)}\, Q \qquad (5.16)$$

where

$$m_{1,2}^2 = {}^1/_2\,(C_{12}^2 + 1)\,p^2 + 3C_{32}^2) \pm \sqrt{{}^1/_4\,(C_{12}^2 + 1)\,p^2 + 3C_{32}^2)^2 - C_{12}^2\,(p^4 + 3p^2)}$$

An expansion of $p^2 u_0^L$ in a series in negative powers of p in the neighborhood of the point at infinity results in the following

$$Q^{-1} p^2 u_0^L = 1 + \frac{0.110}{p^2} - \frac{0.191}{p^4} + \frac{0.399}{p^6} - \frac{0.892}{p^8} + \frac{1.92}{p^{10}} + \ldots \qquad (5.17)$$

Hence (5.18)

$$Q^{-1} u_0^{\cdot} = 1 + 0.0550\tau^2 - 0.00795\tau^4 + 0.553\cdot 10^{-3}\tau^6 - 0.221\cdot 10^{-4}\tau^8 + 0.530\cdot 10^{-6}\,\tau^{10} + \ldots$$

Graphs corresponding to Formulas (5.13),(5.15) and (5.18) and given in Fig.4 determine the power consumed by the plate and give a representation of the possibilities of various simplified equations for small τ.

The relation between the longitudinal force and the mean velocity over the cross section initially corresponds to a one-dimensional deformation and then to a plane state of stress ($\sigma_{zz} = 0$). This transition is accompanied by oscillation relative to the asymptotic ($\tau \to \infty$) value. Equations (3.2) satisfactorily describe this process. The equations of the second order (5.12) and (5.14) determine the initial dependence (first) and the asymptotics (second).

An investigation of flexural deformations is made difficult by the fact that in order to take into account displacements corresponding to self-equilibrating stresses one must solve an equation that is not lower than sixth order (whereas, in the previous case, the second approximation corresponded to an equation of fourth order).

The conclusions obtained above on the character of the propagation of stresses σ_{xx}, symmetrically distributed with respect to the center of the plate, are also true in a number of cases for the propagation of flexural stresses.

Here, however, there is an essential peculiarity, namely, that in bending moment propagation a zone, containing a certain fixed part of the energy (close to total energy), does not widen uniformly, as in the previous case, but decelerates (for $\tau \to \infty$ proportional to $\sqrt{\tau}$). In this connection, the deformations in an interval between the front $x = \tau$ and the indicated zone decrease. It is of interest to clarify how the transition of the wave (formation of the quasi-front) occurs under these conditions and to evaluate in the same sense the velocity c_3 in the case of flexural deformation.

6. Investigation of the equations of dynamic flexure. We take advantage of the fact that for small τ, as well as for a sufficiently large neighborhood of the front for large τ, the mean displacement $0.5w_0$ does not have a decisive influence on the magnitude of bending moment.

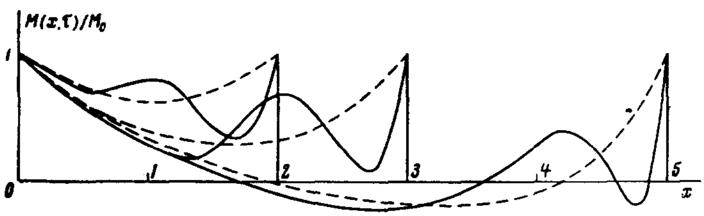

Fig. 5

To convince oneself of this, it is sufficient to compare Fig.5 (dotted line) which gives the distribution of the bending moment $M^* = M(x/\tau)/M_0$ for $0.5 \leqslant \tau \leqslant 5$ (the solid line with account taken of u_1 and w_2, and the dotted line for $w_2 \equiv 0$), with Fig.6 of [17]. We retain in the system (2.7) the quantities u_1 and w_2 (the second approximation for $w_0 = 0$) and set

$$q_x = 3M_0 z\delta_1(x)\,\delta_0(\tau) \qquad (q_z = 0) \tag{6.1}$$

In this case

$$q_{x1} = 2M_0\delta_1(x)\,\delta_0(\tau) \qquad (q_{z2} = 0)$$

We obtain Equations

$$u_1'' - u_1^{\cdot\cdot} - 3C_{21}^2 u_1 + 5(1 - 2C_{21}^2)\,w_2' = -2M_0\delta_1(x)\,\delta_0(\tau) \tag{6.2}$$
$$-3(1 - 2C_{21}^2)\,u_1' + C_{21}^2 w_2'' - w_2^{\cdot\cdot} - 15\,w_2 = 0$$

After some computations similar to those given in Section 3, we obtain the following solution of Equations (6.2) in the form of a series with a variable expansion interval. The bending moment is

$$M(x,\tau) = \sigma_{xx1} = u_1' + 5\left(1 - 2C_{21}^2\right)w_2 = \qquad (0 < x < \tau)$$
$$= M_0 + 2M_0\sum_{n=1}^{\infty}\left\{\frac{1}{\tau}\,\frac{q}{p_1^2 - p_2^2}\left[\cos p_2\tau - \cos p_1\tau + \left(C_{21}^2 q^2 + 60C_{21}^2\left(1 - C_{21}^2\right)\right)\times\right.\right.$$
$$\left.\left.\times\sum_{i=1}^{2}(-1)^i\frac{1-\cos p_i\tau}{p_i^2}\right] - \frac{1-(-1)^n}{\pi n}\right\}\sin qx \quad \left(q = \frac{n\pi}{\tau}\right) \tag{6.3}$$

$$p_{1,2}^2 = {}^1/_2[(1 + C_{21}^2)\,q^2 + 15 + 3C_{21}^2] \pm \{{}^1/_4\,[(1 + C_{21}^2)\;q^2 + 15 + 3C_{21}^2]^2 -$$
$$- C_{21}^2 q^4 - [3C_{21}^4 + 60C_{21}^2(1 - C_{21}^2)\,]q^2 - 45\,C_{21}^2\}^{1/2}$$

Graphs constructed in accordance with (6.3) are shown in Fig.5. Likewise, for comparison, the graph of $M^*(x, \tau)$ for the first approximation is shown (also for $w_0 \equiv 0$). In this case, setting the quantity $w_0 \equiv 0$ in the first of Equations (6.2), we obtain

$$u_1'' - u_1^{\cdot\cdot} - 3C_{21}^2 u_1 = -2M_0 \delta_1(x)\,\delta_0(\tau) \tag{6.4}$$

Hence by means of integral transforms one may find (as in (4.3))

$$M(x, \tau) = u_1' = M_0 \sqrt{3}\, C_{21}\, x \int_x^{\tau} (\tau^2 - x^2)^{-1/2} J_1\left(C_{21} \sqrt{3(\tau^2 - x^2)}\right) d\tau \tag{6.5}$$

in a form of a series

$$M(x, \tau) = M_0 + 2M_0 \sum_{n=1}^{\infty} \left\{ \frac{q}{\tau} \frac{1 - \cos \tau \sqrt{q^2 + 3C_{21}^2}}{q^2 + 3C_{21}^2} - \frac{1 - (-1)^n}{\pi n} \right\} \sin qx$$

$$(0 < x < \tau, \ q\tau = n\pi) \tag{6.6}$$

As is seen from Fig.5, a quasi-front is likewise formed in the case of bending deformations. However, as a consequence of reasons given above, the magnitude of the bending moment in the region of the quasi-front decreases with time.

An exact analysis of the applicability of approximate equations for small τ can be carried out by a comparison of dynamic flexibility determined by the approximate equations and by the theory of elasticity, in the same way as was done in Section 5 for the case of nonflexural deformation of the plate.

We now will determine the dynamic flexibility (the mean rotation angle is $\varphi = {}^3/_2 u_1$) of an infinite plate under the action of a bending moment. We change the right-hand side of the first of the equations of system (5.1) to $-(3/p)M_0 z$.

Then

$$M_0^{-1} \frac{3}{2} u_1^L = M_0^{-1} \varphi^L = \frac{3}{2} \frac{1}{p^2} + \frac{9}{\pi} \left(C_{12}^2 - 2\right) \frac{1}{p} \times$$

$$\times \int_0^{\infty} \left\{ \frac{q^2}{n_1^2} \left[\left(q^2 + n_2^2\right) \left(\cosh n_1 \cosh n_2 - \cosh n_2 \frac{\sinh n_1}{n_1}\right) - 2\left(q^2 - \frac{c_1^2}{c_1^2 - 2c_2^2} p^2\right) \times \right.\right.$$

$$\left. \times \left(\cosh n_1 \frac{\sinh n_2}{n_2} - \frac{\sinh n_1}{n_1} \frac{\sinh n_2}{n_2}\right) \right] + \frac{1}{n_2^2} \left(q^2 - \frac{c_1^2}{c_1^2 - 2c_2^2} p^2\right) (q^2 + n_2^2) \times$$

$$\left. \times \left(\frac{\sinh n_1}{n_1} \cosh n_2 - \frac{\sinh n_1}{n_1} \frac{\sinh n_2}{n_2} \right) - 2q^2 \left(\cosh n_1 \cosh n_2 - \cosh n_1 \frac{\sinh n_2}{n_2}\right) \right\} \frac{dq}{B^*(p^2, q^2)} \tag{6.7}$$

$$\left(B^*(p^2, q^2) = (q^2 + n_2^2)^2 n_1 \sinh n_1 \cosh n_2 - 4q^2 n_1^2 n_2 \cosh n_1 \sinh n_2\right) \tag{6.8}$$

Proceeding in the same way as in the previous case, we find that

$$M_0^{-1} \varphi^{\cdot} = {}^3/_2 + a_1 \tau + a_2 \tau^2 + a_3 \tau^3 \quad \text{for } \tau < 2 \tag{6.9}$$

$$a_1 = \frac{9}{\pi} \int_0^{\infty} \frac{1.96 q^2 dq}{(q^2 + 1)^{3/2} F(q^2)} = 0.255, \qquad a_3 = \frac{3}{2\pi} \int_0^{\infty} \frac{(3.4 - 1.4q^2)^2 dq}{(q^2 + 3.4)^{3/2} (q^2 + 1)^2 F(q^2)} = 0.0365$$

$$a_2 = \frac{9}{\pi}\int_0^\infty \left(\frac{6.76q^2+3.4}{(q^2+1)^2(q^2+3.4)} - \frac{4.8}{(q^2+1)^{3/2}\sqrt{q^2+3.4}}\right)\frac{q^2dq}{F(q^2)} = -0.383 \quad (6.10)$$

$$\left(F(q^2) = (2q^2+3.4)^2 - 4q^2\sqrt{q^2+1}\sqrt{q^2+3.4}\right)$$

A graph of the angular velocity $\varphi^{\cdot}$ (6.9) is shown in Fig.6 (curve 1), where the quantity ${}^2/_3\ \varphi^{\cdot} M_0^{-1}$ is given on the ordinate.

We consider now how the simplified equations describe the dynamic flexibility of a model of the plate.

1) The equations of bending deformations taking into account w_0, u_1. The basic operator (2.11)

$$C_{21}^2 w_0'' - w_0^{\cdot\cdot} + 3C_{21}^2 u_1' = -q_{z0} = 0 \quad (6.11)$$

$$-C_{21}^2 w_0' + u_1'' - u_1^{\cdot\cdot} - 3C_{21}^2 u_1 = -q_{x1} = -2M_0\delta_1(x)\,\delta_0(\tau)$$

After an application of the Laplace transform we obtain for $x = 0$

$$\frac{\varphi^L}{M_0} = \frac{3}{2}\frac{u_1^L}{M_0} = \frac{3}{2}\frac{1}{p}\frac{m_1m_2 + C_{12}^2p^2}{m_1m_2(m_1+m_2)} \quad (6.12)$$

$$\left(m_{1,2}^2 = {}^1/_2\left(C_{12}^2p^2 \pm \sqrt{{}^1/_4(C_{12}^2-1)^2p^4 - 3p^2}\right)\right)$$

An expansion (6.12) in negative powers of p in the neighborhood of the point at infinity is the following (6.13)

$$M_0^{-1}p^2\varphi^L = {}^3/_2 - 0.658\alpha + 0.492\alpha^2 - 0.410\alpha^3 + 0.359\alpha^4 - 0.323\alpha^5 + \ldots$$

$$(\alpha = 0.882\ p^{-2})$$

Thus

$$M_0^{-1}\varphi^{\cdot} = {}^3/_2 - 0.329\tau^2 + 0.0205\tau^4 - 0.569\cdot10^{-3}\tau^6 + $$
$$+ 0.889\ 10^{-5}\tau^8 - 0.890\cdot10^{-7}\tau^{10} + \cdots \ (\tau > 0) \quad (6.14)$$

2) The Timoshenko equations (2.19). We note that after the substitution

$$\tau = (1.2)^{1/2}\tau_*$$

and multiplication of the right-hand sides by the factor 1.2 Equations (6.11) practically do not differ from the Timoshenko equations (2.19). Hence the Timoshenko equations determine the angular velocity (6.15)

$$\varphi_*^{\cdot} = \sqrt{1.2}\,\varphi^{\cdot}(\tau/\sqrt{1.2})$$

Here $\varphi^{\cdot}(\tau)$ is determined by Expression (6.14)

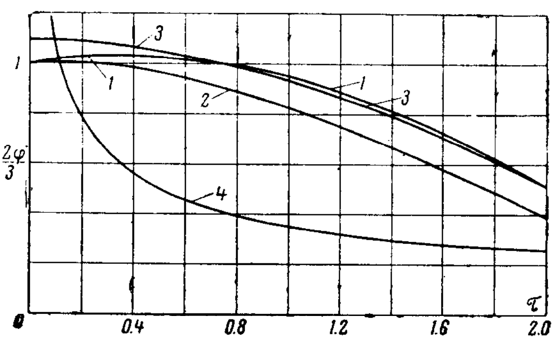

Fig. 6

3) The Bernoulli-Euler equation. This equation differs from the equations of Timoshenko in that the following are disregarded: the longitudinal inertia $(u_1^{\bullet\bullet} \equiv 0)$, the warping of the cross section $(u_3 \equiv 0)$ and the shear $(3u_1 = -w_0'$, but $\sigma_{xz0} \not\equiv 0!)$

$$^1/_3 C_{31}^2 w_0'''' + w_0^{\bullet\bullet} = q_{z0} + q_{x1}' \qquad (6.16)$$

Therefore for $q_{z0} = 0,\ q_{x1} = 2M_0\delta_1(x)\,\delta_0(\tau)$

$$\varphi^L = \frac{1}{2} w_0'^L(0,\ p) = M_0 \frac{0.930}{p^{3/2}}, \qquad \varphi^{\bullet} = M_0 \frac{0.525}{\sqrt{\tau}} \qquad (6.17)$$

Graphs constructed in accordance with Formulas (6.14), (6.15) and (6.17) are shown in Fig.6 (curves 2, 3, 4, respectively).

The deformations $\partial w/\partial z$ also increase the dynamic flexibility in the case of the action of a bending moment. As a result of this, Equations (6.11), which do not take into account the components w_n for $n > 0$, give an accurate result only at the very beginning of the process, and thereafter give a certain error.

At the very beginning, the result corresponding to the Timoshenko differs from the exact result by 10%, but thereafter very rapidly nears the result of the theory of elasticity (in the time of propagation of the dilatation wave over one quarter of the thickness of the plate). Apparently the equations of the sixth order (*), which take into account the components w_0, u_1, w_2 and the asymptotic value (as in Timoshenko equations) of the components u_3, determine the angular velocity for all τ practically exactly.

The Bernoulli-Buler equation for small τ (in the region shown on the graphs) is not applicable.

These results allow one to make the following assertions.

1. The Saint-Venant principle is applicable for the study of transient processes in beam dynamics since deformations corresponding to suddenly applied self-equilibrating loads localize themselves in the neighborhood of the wave fronts and in the neighborhood of the cross section over which the load is applied.

2. This assertion does not extend to self-equilibrating disturbances with the continuous inflow of energy into the beam (for example, to periodic disturbances).

3. The generalization (in the above-indicated sence) of the Saint-Venant principle to beam dynamics gives the possibility of studying not only slowly changing but also rapidly changing transient processes by means of the equations of longitudinal and flexural oscillations (2.16) and (2.17). The reasons here are typically the same as those that allow one to apply the elementary static theory of beams and plates in design in the presence of concentrated loads. The solution which is obtained thereby gives a correct representation of the propagation of energy along the beam and of the change of the non-self-equilibrating stresses over the cross section.

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*) For the displacement problem in the presence of transverse loads, the best sixth-order equations are those that take into account the components w_0, u_1, u_3 and the asymptotic value of the component w_2.

4. The constant c_3 in (2.16) determines the velocity of propagation of the wave front for longitudinal disturbance of the beam. In essence, however, it is the velocity of propagation of the quasi-front. In the approximate equation (2.16), the deformations in the region between the quasi-front and the actual front are neglected.

The constants v_1, v_2 in (0.2) determine the velocity of propagation of discontinuities of a flexural disturbance of the beam. However, v_1, v_2, which do not coincide with the actual velocity of propagation of discontinuities c_1, c_2, may be interpreted as velocities of propagation of the quasi-fronts only in the very first period of motion ($\tau <$ 5 to 7). Further, the quasi-fronts essentially vanish, after which it is not possible to connect with v_1, v_2 the propagation of any characteristic singularities of bending deformations. Hence v_1, v_2, perhaps, are most correctly interpreted as velocities c_1, c_2 distorted as a result of the approximations contained in the arguments with which (0.2) were derived. It is interesting, however, that although the identification of v_1, v_2 with c_1, c_2 gives a correct description of the propagation of discontinuities, it nevertheless gives a less accurate description of the general picture of the deformation of the beam, and, in particular in determining its dynamic flexibility (Section 6) than for example if v_1, v_2 are determined by the recommendation of Timoshenko.

5. Although Equations (2.16), (2.17) do not give a jump in the stresses in the regions of the actual fronts, this does not lead to any essential error. The reason is that the peak of stresses in the region of the front shrinks rapidly and, as a result of this, in actual cases, when the loads are not instantaneously applied, the magnitude of the stresses in the neighborhood of the front will decrease. An analysis of the influence of the rate of loading on the magnitude of the stresses at the front was carried out on a model consisting of two parallel beams elastically connected [18].

In the study of transient beam dynamics one may use, in addition to Equations (0.2) and (2.17), the equation of Bernoulli-Euler (0.1) which beginning with $\tau = 7$ to 10 gives already a solution close to the solution of (0.2) [19]. The circumstance that the velocity of propagation of discontinuities according to (0.1) is infinite, does not turn out to have an essentially quantitative effect on the deformation of the neutral axis and on the picture of the propagation of the energy along the beam.

The advantage of the "wave" equation (0.2) compared to the "nonwave" equation (0.1) is the possibility of clarifying the beam deformation picture for $\tau < 7$, whereby, as is seen from Section 6, the correct evaluation of the dynamic flexibility is obtained down to $\tau = 0$.

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